NEWTONIAN FLOW THEORY IN HYPERSONIC AERODYNAMICS

By Wallace D. Hayes

Princeton University and Space Technology Laboratories

Summary—A brief description is given of the body of theory known as Newtonian flow theory, the theory based on the assumption that the shock layer in a hypersonic flow is infinitesimally thin. The reasons why the study of this theory is essential to an understanding of hypersonic aerodynamics are emphasized.

The accomplishments of Sir Isaac Newton in gas dynamics are discussed, together with the development of the pressure laws referred to as Newtonian. Some of the features of the general theory are discussed, including optimum shapes for minimum drag and solutions with cross flow.

In the field of hypersonic aerodynamics the phrase "Newtonian pressure law" or some equivalent appears with great frequency. Closely related is a body of aerodynamic theory also referred to as Newtonian, which we term "Newtonian Flow Theory". This theory is based upon the assumption that the density in the shock layer of a hypersonic flow is very much greater than the free stream density, and accordingly that the shock layer is extremely thin. A development of Newtonian flow theory will appear in Chapter III in a forthcoming book by Hayes and Probstein(1), and it is on this source that this paper is based. Our purpose will be to give a brief review of this theory and of its relation to hypersonic aerodynamics.

The basic parameter which is most important in assessing the applicability of Newtonian flow theory is the ratio of the density in the free stream ρ_∞ to the density immediately behind the shock ρ_s. Thus we define:

$$\epsilon = \rho_\infty / \rho_s \tag{1}$$

Although ϵ is not a constant for a curved shock wave, it is convenient to consider it as though it were a single parameter. For Newtonian flow theory to be generally applicable the quantity $\epsilon^{1/2}$ must be small compared with 1. Even in air at high temperature ϵ does not become much less than about 1/15, and we must accept the fact that Newtonian flow theory cannot yield accurate results for actual hypersonic flows.

The obvious question arises as to why, in light of the inapplicability of this theory to practical flows, it should be of any interest in hypersonic aerodynamics. A number of reasons appear for this interest. The quantity ϵ is a basic parameter in hypersonic aerodynamics, and to really understand its role we must understand the Newtonian limiting case $\epsilon \to 0$. Various fruitful theoretical approaches to problems of hypersonic flow are based through a successive approximation scheme on the smallness of the parameter ϵ. In these approaches the Newtonian theory gives the lower-order approximation. Newtonian flow theory has a considerable instructional value. For example, it presents us with numerous anomalies and examples of singular behavior, each of which must have a corresponding vestige in flows with ϵ small but finite. And finally, we should not overlook the not-too-unlikely possibility that we may be confronted with physical flows with shock waves for which ϵ is truly very small.

We turn now to the work of Sir Isaac Newton, which appears in Book II of his *Principia Mathematica*(2). In Newton's time the sciences of thermodynamics and of kinetic theory had not yet been invented, and any development of gas dynamics in its present-day sense would have been impossible. But Newton did make very significant deductive contributions in a field which, in all fairness, must be included in gas dynamics. He created a model of a gas which obeys Boyle's Law, in which the elasticity of the gas is provided through repulsive forces acting between every two neighboring particles which vary inversely as the distance between them (Proposition 23). With this model Newton establishes conditions for the complete similarity of the motion of two different systems (Proposition 32). The principal condition is that the ratio $p/\rho v^2$ must be the same in both systems, where p, ρ, and V are characteristic values of the pressure, density, and velocity. This condition is equivalent to the condition that the two systems have the same characteristic Mach number, and this Newtonian similarity is equivalent to Mach number similarity (cf. Cranz [Ref. 3, p. 45]).

Newton also establishes the result that if the velocities are high enough the repulsive interparticle forces (and the free stream pressure) may be neglected, and that in the resulting flow the resistance varies as the square of the velocity, accurately (Proposition 33, Corollaries I and II). In this result Newton has the equivalent of what we term the "Mach number independence principle", of the similitude expressed by Oswatitsch(4).

Newton also studied the case of the flow of a very rarefied gas impinging on a solid body, and it is from this part of his work that the Newtonian pressure law originates. In Newton's model for this flow, the only forces appearing in the problem were the forces of impact of individual particles with the body. Newton considered the case of specular reflection (Case I) and the case in which all the normal momentum of a particle is transferred to the body and the tangential momentum of the particle is conserved (Case II). Although his Case II gives realistic estimates for the pressure force on a cold body in the flow of a rarefied gas, it cannot be considered realistic because it provides for no shear stress. In either case the pressure on the body is proportional to $\sin^2 \theta$, where θ is the inclination angle of the body surface with respect to the free stream. In classical exterior ballistics (cf. Cranz [Ref. 5, § 12]) this pressure law is presented empirically, with an unknown multiplicative constant.

The result corresponding to Newton's Case II is:

$$\tfrac{1}{2}C_p = \sin^2\theta \tag{2}$$

and with modifications is in current use today in hypersonic aerodynamics. This result was first derived for an inviscid continuum gas flow by Epstein[6], for the case of a wedge with ϵ very small. In this case Newton's assumption for momentum transfer in his Case II are borne out, so that Newton's analysis becomes correct. The result is not correct for the pressure on a curved body, as was pointed out by Busemann [Ref. 7, p. 275–7], because of the centrifugal forces necessary to make the thin shock layer follow a curved path. The correct result is the Newton–Busemann pressure law:

$$\tfrac{1}{2}C_p = \sin^2\theta - KP, \tag{3}$$

where K is the curvature of the body and $\rho_\infty U^2 P$ is the momentum flow in the shock layer per unit depth.

The Newtonian pressure law (2) without the Busemann correction is often used today either to estimate pressures on bodies or to serve as a basis for comparisons of experimentally measured pressures on bodies. It should be strongly emphasized that the pressure law (2) without the Busemann correction, with or without other modifications, has no theoretical basis whatsoever for curved bodies. It does, indeed, have a reasonably good empirical basis, but it must be recognized that its only basis is an empirical one.

In Newtonian flow theory as applied to two-dimensional bodies, the momentum flow term P is computed on the basis that the velocity within the shock layer is constant along each streamline. For a streamline which enters the layer at a given value of the lateral coordinate y for which the inclination angle is $\theta(y)$, this velocity is $U\cos\theta$. With the differential element of mass flow per unit depth equal to $\rho_\infty U\,\mathrm{d}y$, we may express:

$$P_1 = \int_0^{y_1} \cos\theta\,\mathrm{d}y \tag{4}$$

for P evaluated at $y = y$. The streamline $y = 0$ in the free stream is taken to be the dividing streamline (see Fig. 1).

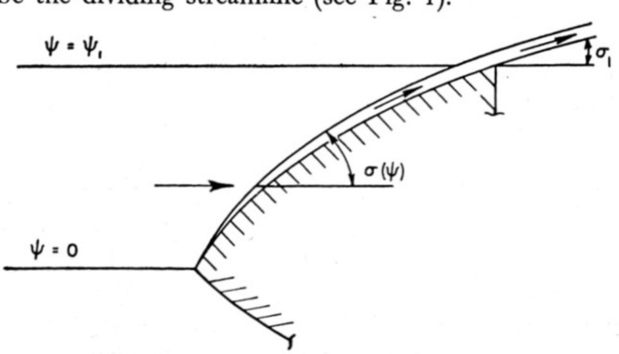

FIG. 1. Newtonian flow.

The drag per unit depth D of a two-dimensional half-body may be calculated most easily by means of a momentum balance, and may be expressed:

$$\frac{D}{\rho_\infty U^2} = y_1 - P_1 \cos \theta_1, \tag{5}$$

where the subscript 1 indicates the rear edge of the body or part of the body being considered.

The corresponding lift per unit depth is:

$$\frac{L}{\rho_\infty U^2} = \pm P_1 \sin \theta_1. \tag{6}$$

An important concept in Newtonian flow theory is that of the free layer. With the curvature K sufficiently high and the angle θ sufficiently low, the pressure as given by (3) may become zero and would become negative if the flow were required to follow the body. We disregard the possibility of negative pressure and are led to the concept of a Newtonian shock layer which separates from the body and flies free. The shape of this free layer is determined by the condition that the pressure behind the layer is zero (see Fig. 2).

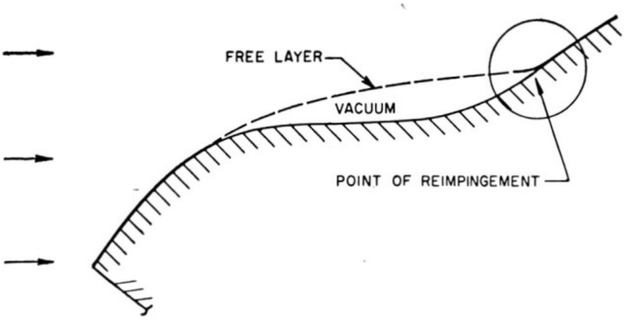

FIG. 2. Free layer.

With the pressure set equal to zero, (3) may be readily integrated to yield the shape of the free layer. The shape appears as a parabola with its vertex pointed upstream in two-dimensional flow. In an axi-symmetric flow the shape is a cubic curve. The concept of the free layer may be generalized to include constant-pressure layers and sails.

We turn now to the question of obtaining body shapes which are optimum in the sense of having minimum drag. For simplicity we restrict ourselves to the case of a two-dimensional half-body with given width and length, but without additional isoperimetric conditions. In (5), we note that the term y_1 is the body width and is fixed. Minimum drag is obtained by maximizing the product $P_1 \cos \theta_1$. We now make the assumption that we may maximize each factor separately, and we set $\theta_1 = 0$, $\cos \theta_1 = 1$. By the calculus of variations, maximum P_1 is obtained with a straight line for the body shape, and we have the apparently inconsistent result that the inclination angle θ is *not* zero at the trailing edge. The fundamental inconsistency vanishes, however, when we note that mathematically we may have a discontinuity in the quantity θ at the trailing edge.

Such a discontinuity in slope, though permitted mathematically, leads to the unacceptable physical requirement of a negative delta function in pressure at the trailing edge. This difficulty is resolved by the introduction of the concept of a "Newtonian thrust cowl", a small cowling at the trailing edge which exerts a positive delta function in pressure on the outside of the shock layer in order to turn it into the final direction

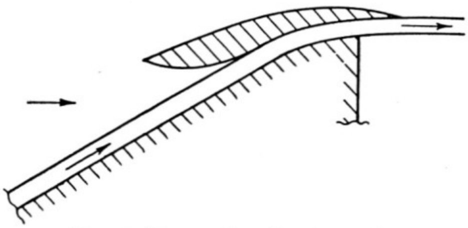

FIG. 3. Newtonian thrust cowl.

$\theta_1 = 0$ (see Fig. 3). Optimum shapes of this type are termed "absolute optimum" shapes.

Shapes which are optimum under the additional restrictions of no Newtonian cowl and non-negative pressure on the body are termed "proper optimum" shapes. For the simple case we are considering the proper optimum shape consists of a forebody with a maximum P shape (a straight-line segment), followed by an afterbody on which the pressure is zero.

A free layer erupts from the end of the forebody and just grazes the trailing edge at the rear of the afterbody (see Fig. 4). For the two-dimensional body the chord of the forebody is 1/2 the chord of the total body.

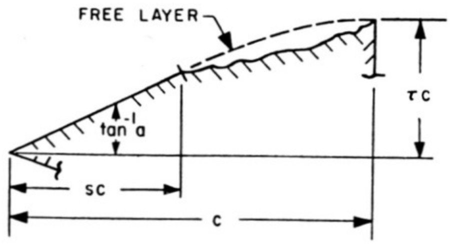

FIG. 4. Proper optimum body.

For the corresponding slender body of revolution the shape is a 3/4 power shape and the chord ratio is 3/5.

The drags obtained with absolute and proper optimum shapes are given in the Tables 1 and 2 for slender two-dimensional and axisymmetric shapes. One conclusion from the Newtonian theory which should be general in hypersonic aerodynamics is that optimum bodies are not obtained with simple power-law shapes.

If cross-flow is present, an analysis is necessary which takes the individual particle trajectories into account. A particle on striking the surface takes the "fall line" direction, the direction of steepest descent if the free stream flow is taken to be directed vertically downward. Subsequently, it follows at constant velocity a geodesic path, a path of zero lateral curvature on the body surface. For the analysis of solutions with cross flow an essential concept is that of the "locus of entering streamlines", the locus of points on the body such that a particle entering at such a point passes over a particular other point being investigated on the body (see Fig. 5).

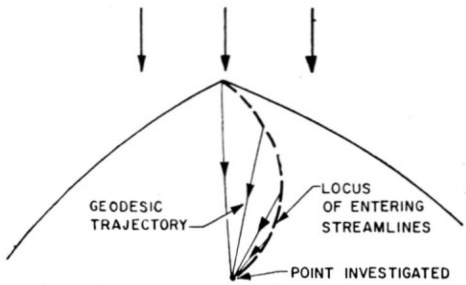

FIG. 5. Locus of entering streamlines.

For the pressure on the body a formula similar to (3) may be obtained. The term KP of (3) is replaced by the double dot product between the two-dimensional symmetric tensor or dyadic representing the curvature of the body and a two-dimensional symmetric tensor or dyadic representing the momentum flow in the shock layer. To obtain this momentum tensor the structure of the shock layer must be known.

This cross-flow analysis may be carried out in principle for any three-dimensional body shape, and may be carried out in closed form in certain simple cases. The method is of interest in hypersonic aerodynamics because it may be applied to problems in which the restrictions of Newtonian theory are relaxed and which may be considered to be more realistic, and because of the conceptual understanding obtainable from the study of the Newtonian theory.

TABLE 1

Drags of slender two-dimensional shapes

Shape	D/D_{wedge}
Wedge	1·000
0·864 Power (J. Cole)	0·918
Proper optimum	0·770
Absolute optimum	0·500

TABLE 2

Drags of slender axi-symmetrical shapes

Shape	D/D_{cone}
Cone	1·000
3/4 power	0·703
2/3 power (J. Cole)	0·667
Proper optimum	0·576
Absolute optimum	0·422

REFERENCES

1. W. D. HAYES and R. F. PROBSTEIN, *Hypersonic Flow Theory*, Academic, 1959.
2. I. NEWTON, *Mathematical Principles of Natural Philosophy*, transl. by A. MOTTE (1729), revised by A. Cajori, University of California Press, Berkley, California, 1934; reprinted in 1946.
3. C. CRANZ, *Lehrbuch der Ballistik*, *Ergänzungsband*, J. Springer, Berlin, 1936. Reprinted by Edwards Brothers, Ann Arbor, Michigan, 1943.
4. K. OSWATITSCH, *Z. angew. Math. Phys.*, Vol. 2, pp. 249–64, 1951.
5. C. CRANZ, *Lehrbuch der Ballistik*, Vol. I (*Äussere Ballistik*), 5th edition. J. Springer, Berlin, 1925. Reprinted by Edwards Brothers, Ann Arbor, Michigan, 1943.
6. P. S. EPSTEIN, *Proc. Nat. Acad. Sci. U.S.A.*, Vol. 17, pp. 532–47, 1931.
7. A. BUSEMANN, *Handwörterbuch der Naturwissenschaften*, Vol. 4, second edition. Gustav Fischer, Jena, 1933.